

Sparsification and subexponential approximation

Édouard Bonnet Vangelis Th. Paschos*
 PSL Research University, Université Paris-Dauphine
 LAMSADE, CNRS UMR 7243, France
 edouard.bonnet@dauphine.fr, paschos@lamsade.dauphine.fr

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Abstract

Instance sparsification is well-known in the world of exact computation since it is very closely linked to the *Exponential Time Hypothesis*. In this paper, we extend the concept of sparsification in order to capture subexponential time approximation. We develop a new tool for inapproximability, called approximation preserving sparsification and use it in order to get strong inapproximability results in subexponential time for several fundamental optimization problems as MAX INDEPENDENT SET, MIN DOMINATING SET, MIN FEEDBACK VERTEX SET, and MIN SET COVER.

1 Introduction

The most common way to cope with intractability in complexity theory is the design and analysis of efficient approximation algorithms. The main stake of such algorithms is to “fastly” compute feasible solutions for the hard problems tackled (avoiding so, if possible, long and time-consuming computations needed for determining optimal solutions). The values of these solutions must be as “close” as possible to the optimal values.

Historically, the first research program dealing with approximation, was the *polynomial time approximation theory* founded back in 1974 with the seminal paper [1]. Since the early 90’s, using the celebrated PCP theorem ([2]), numerous natural hard optimization problems have been proved to admit more or less pessimistic inapproximability results. For instance, for any $\epsilon > 0$, MAX INDEPENDENT SET is inapproximable within approximation ratio $n^{\epsilon-1}$, unless $\mathbf{P} = \mathbf{NP}$ ([3]). Similar results, known as *inapproximability* or *negative results*, have been provided for numerous other paradigmatic optimization problems.

To remedy to this pessimistic context, two complementary research programs, dealing with super-polynomial approximation, came to be added in the approximation landscape. The first one, called *parameterized approximation*, handles approximation by fixed parameter algorithms. This line of research was initiated by three independent works [4, 5, 6]. The second research program, called *moderately exponential approximation*, seeks, given a problem Π , for r -approximation algorithms with running time significantly faster than those of exact algorithms computing optimal solution for Π . This issue has been independently developed by [7, 8, 9, 10].

However, a fundamental question remained globally unanswered by both of them. Is *subexponential* approximation possible for some paradigmatic optimization problems as,

*Also, Institut Universitaire de France

for instance, MAX INDEPENDENT SET, MIN VERTEX COVER, or MIN DOMINATING SET? A first answer about MAX INDEPENDENT SET and MIN VERTEX COVER has been provided in [11] where it is proved the following.

Theorem 1. [11] *Under ETH¹, in graphs of order n :*

1. *for any positive constant r and any $\delta > 0$, there is no r -approximation algorithm for MAX INDEPENDENT SET running in time $O^*(2^{n^{1-\delta}})$;*
2. *for any $\epsilon > 0$ and any $\delta > 0$, there is no $(7/6 - \epsilon)$ -approximation algorithm for MIN VERTEX COVER running in time $O^*(2^{n^{1-\delta}})$.*

The result of Item 1 of Theorem 1 has been powerfully improved by [13], where a very clever implementation of PCP [14] leads to the following theorem.

Theorem 2. [13] *Under ETH, in graphs of order n with maximum degree Δ :*

1. *(General graphs) for any $\delta > 0$ and any r larger than some constant, any r -approximation algorithm for MAX INDEPENDENT SET runs in time at least $O^*(2^{n^{1-\delta}/r^{1+\delta}})$;*
2. *(Δ -sparse graphs²) for any sufficiently small $\epsilon > 0$, there exists a constant Δ_ϵ , such that for any $\Delta \geq \Delta_\epsilon$, MAX INDEPENDENT SET on Δ -sparse graphs is not $\Delta^{1-\epsilon}$ -approximable in time $O^*(2^{n^{1-\epsilon}/\Delta^{1+\epsilon}})$.*

Our goal in this paper is to introduce a new technique based upon the development of a novel notion of *approximation preserving sparsification* that extends the scope of the classical sparsification of [12]. Then, using approximation preserving sparsifiers, we derive negative results for MAX INDEPENDENT SET in bounded degree graphs as well as for several fundamental problems as MIN DOMINATING SET, MIN FEEDBACK VERTEX SET, etc.

2 Preliminaries

The idea of *instance sparsification* (with respect to some parameter) has been introduced in [12] and is very closely related to the ETH. Informally, starting from an instance ϕ of k -SAT, with n variables and m clauses, the sparsification of [12] consists of building $2^{\epsilon n}$ (for some constant $\epsilon > 0$) “sparse” instances for the problem, i.e., formulæ on n variables and δn clauses, for some $\delta > 0$, such that ϕ is satisfiable if and only if one of the sparse formulæ is satisfiable. Let us note that the sparsification of [12] *is not approximation preserving*. One of the reasons for this, is that when a clause C has all its literals contained in a clause C' , a reduction rule removes C' , that is safe for the satisfiability of the formula (hence, for exact computation), but not for approximation.

When handling graph problems (or problems that can be represented by means of a graph; this is, for example, the case of MIN SET COVER), a natural parameter upon which one can apply sparsification is the maximum degree Δ of the input graph. So, a natural sparsification schema for such problems is to start from a graph G of order n with arbitrarily large Δ and to produce a large number of graphs G_i ’s of order bounded by n and whose maximum degree Δ' is bounded by “something” smaller than Δ and such that some solution with a proved ratio for one G_i can be transformed into a solution with at

¹The *Exponential Time Hypothesis* (ETH) [12] postulates that there exists an $\epsilon > 0$ such that no algorithm solves 3-SAT in time $O^*(2^{\epsilon n})$, where n is the number of variables. This is a widely-acknowledged computational assumption.

²Graphs where the maximum degree is bounded by Δ .

least the same ratio for G . Consider an instance G (of size n) of an optimization problem Π and denote by Δ the degree of G . Let $\Pi\text{-}B$ denote the problem Π restricted to graphs with degree at most B . Informally, an *approximation preserving sparsification* from Π to $\Pi\text{-}B$, maps G into a set $\{G_1, G_2, \dots, G_t\}$ of subgraphs of G and maps a solution S_i of G_i into a solution S of G , this latter transformation taking polynomial time; $t \leq 2^{\epsilon n}$, for some $\epsilon > 0$, and G_i 's are such that any of them has degree at most B_ϵ , for a constant B_ϵ independent on n . Furthermore, if some S_i is an r -approximation of $\Pi\text{-}B(I_i)$, then S is an r -approximation in G .

In Section 3 we first formalize the concept of approximation preserving sparsification and then we propose two such sparsifiers. The first sparsifier, called *superlinear sparsifier*, is devised along the line informally described just above and generalizes the (linear) sparsifier introduced in [11]. The superlinear sparsifier, in fact, relaxes the requirement that B_ϵ has to be constant (this was the case of the sparsifier in [11]) and allows the sparsification tree to stop even for non-constant degrees. For simplicity, we present this sparsifier for the case of MAX INDEPENDENT SET and MIN VERTEX COVER, but similar sparsifiers can be developed for several other problems, in particular for the APETH-equivalent problems of [11]. One of the interesting features of this sparsifier is that it allows the transfer of negative results to problems linked to MAX INDEPENDENT SET, or to MIN VERTEX COVER, by approximability preserving reductions building instances of size $O(n + m)$, where m denotes the number of edges of the input graph. The second sparsifier devised in Section 3, is called *k-step sparsifier* and runs in polynomial time. It deals with problems whose solutions satisfy some domination property (as MAX INDEPENDENT SET, MIN DOMINATING SET, MIN INDEPENDENT DOMINATING SET, and MIN VERTEX COVER) and gives quite interesting results when handling maximization problems.

Using either superlinear or k -step sparsifier, together with gap-preserving reductions, we prove in Section 4 rather strong negative subexponential inapproximability results for several fundamental problems. More precisely:

- via *superlinear sparsifier* we show that *under ETH, and for any $\epsilon > 0$, none of MIN DOMINATING SET, MIN SET COVER and MIN HITTING SET, MIN FEEDBACK VERTEX SET, MIN INDEPENDENT DOMINATING SET, and MIN FEEDBACK ARC SET can be $(7/6 - \epsilon)$ -approximable in time $O^*(2^{n^{1-\epsilon}})$* ;
- via *k-step sparsifier* we show that *under ETH, for any $\epsilon > 0$ and any $\Delta < \Delta_\epsilon$, in Δ -sparse graphs and in time $O^*(2^{O(n^{1-\epsilon}/\Delta_\epsilon^{1+\epsilon})})$, MAX INDEPENDENT SET, MAX ℓ -COLORABLE INDUCED SUBGRAPH and MAX INDUCED PLANAR SUBGRAPH are inapproximable within ratios $\Delta/2 - (\Delta_\epsilon/2 - \Delta_\epsilon^{1-\epsilon})$, $\Delta/2 - (\Delta_\epsilon/\ell - \Delta_\epsilon^{1-\epsilon})$ and $\Delta - (\Delta_\epsilon - \Delta_\epsilon^{1-\epsilon})$, respectively*;
- finally, using Item 1 of Theorem 2 we show that *under ETH, for any $\delta > 0$ and any $r \geq n^{1/2-\delta}$, MAX MINIMAL VERTEX COVER and MIN INDEPENDENT DOMINATING SET are inapproximable within ratios $(c+r)/(1+c)$ and $1/(1-c)$ respectively, in less than $O^*(2^{n^{1-\delta}/r^{1+\delta}})$ time, in a graph of order nr , with c the stability ratio of the MAX INDEPENDENT SET-instance of [13]*.

Our technique for proving negative results via approximation preserving sparsification (on graph problems) can be outlined as follows. Let Π be some problem inapproximable in time $O^*(2^{n^{1-\epsilon}})$, for any $\epsilon > 0$, Π' be some problem such that Π reduces to Π' by some approximation preserving reduction R that works in polynomial time and builds instances of Π' of size $n + m$, and let \mathcal{F} be a superlinear approximation preserving sparsifier for Π . Then, for an instance G of Π we do the following:

- apply \mathcal{F} to G in order to build at most $O^*(2^{n^{1-\epsilon}})$ n^ϵ -sparse instances G_i ;
- transform any sparse instance G_i into an instance G'_i of Π' ;
- if Π is not approximable in time $O^*(2^{n^{1-\epsilon}})$ within ratio r and if R transforms any ratio r' for Π' into ratio $r = c(r')$ for some invertible function c , then Π' is no more approximable in time $O^*(2^{n^{1-\eta(\epsilon)}})$ within ratio $c^{-1}(r)$.

In what follows, we use standard notation from graph theory as $\Gamma(v)$, the set of neighbours of vertex v , $G[V']$ the subgraph of G induced by V' . Given a set system (\mathcal{S}, C) , the frequency of the system is defined as the maximum number of subsets an element of C belongs to. Some of the results are given here without proofs. All missing proofs can be found in the appendix.

3 Approximation preserving sparsifiers

We first informally describe the basic idea behind sparsification [12] and its use for deriving lower bounds in exact computation. Assuming a reference problem Π' cannot be solved in $O^*(\lambda^n)$, for some $\lambda > 1$, we are interested in showing that another problem Π cannot be solved in $O^*(f(\lambda)^n)$. For instance, if the reference problem is SAT and $\lambda = 2$, our assumption is the Strong ETH (SETH).

For doing this, we use reductions from Π' to Π . Note that one can easily derive negative results if there exists a linear reduction from Π' to Π (i.e., a reduction with linear instance-size amplification). But, unfortunately, linear reductions are quite rare, so that approach is limited. Yet, reductions where Π' is a graph problem, amplifying the instance to a size $O(n + m)$ where n is the number of vertices and m the number of edges (or, dealing with some satisfiability problem, n is the number of variables, and m the number of clauses) are much less rare.

A way to overcome non-linearity is to “*sparsify*” instances of Π' , producing, from an instance I , $\gamma(n)$ instances where the number of edges is linear to n and to prove that, for at least one of them, an optimal solution is also (or can be transformed in time at most $O^*(\gamma(n))$ into) an optimal solution for I . We then apply the reduction to all of these sparsified instances.

In other words, for the non-linear reductions to produce non-trivial results, we need a not too costly preprocessing step (sparsification) which makes the number of edges (resp., clauses) linear in the number of vertices (resp., variables).

The sparsifier for SAT, presented in [12], shows that *for every integer $k \geq 3$, and every $\varepsilon > 0$ there exists a constant $C_{\varepsilon,k}$ and $2^{\varepsilon n} C_{\varepsilon,k}$ -sparse instances of k -SAT whose disjunction is equivalent to the initial instance.* But, as noticed above this idea does not work for approximation.

In Section 3.1 we extend sparsification to approximation by implementing a sparsifier for a large class of maximisation problems (whose solutions are subsets of the vertex-set of the input graph verifying some property) that works not only for exact computation but also for approximation.

3.1 Superlinear sparsifier

Given an optimization graph problem Π and some parameter of the instance (this can be, for instance, the maximum, or the average degree) let $\Pi-B$ be the problem restricted to instances where the parameter is at most B (we use the same notations as [11]). Then, a superlinear sparsifier can be defined as follows.

Definition 1. An approximation preserving superlinear sparsification from a graph problem Π to its bounded parameter version $\Pi\text{-}B$ is a pair (f, g) of functions such that, given any function ϕ , sublinear in n , and any instance G of Π :

- f maps G into a set $f(G, \phi) = (G_1, G_2, \dots, G_t)$ of instances of Π , where $t \leq 2^{\phi(n)}$ and the orders n_i of the G_i 's are all bounded by n ; moreover, there exists a function ψ (depending on ϕ) such that any G_i has parameter at most $\psi(n)$ (for instance, if the parameter is the degree of the graph, the number of edges of G_i 's is linear in n , if ψ is constant, superlinear otherwise);
- for any $i \leq t$, g maps a solution S_i of an instance $G_i \in f(G, \phi)$ into a solution S of G ;
- there exists an index $i \leq t$ such that if a solution S_i is an r -approximation for G_i , then $S = g(G, G_i, S_i)$ is an r -approximation for G ;
- f is computable in time $O^*(2^{\phi(n)})$, and g is polynomial in $|n|$.

For simplicity, the sparsifier of Definition 1 has been specified in the case of graph problems and assuming that it transfers the same ratio r from the leaves of the sparsification tree to its root. One can easily see that it can be generalized to any constant transfer function.

It is also easy to see that the sparsifier can be easily extended to problems defined on set-systems, as MIN SET COVER MIN HITTING SET, or MAX SET PACKING. Here, parameters can be the cardinality of the largest set, or the frequency. It can also be extended to fit optimum satisfiability problems, where as parameter B can be considered the maximum occurrence of a variable in the input formula. The soundness of this sparsifier relies on the following folklore lemma.

Lemma 1. An algorithm with branching vector $(1, \psi(n))$ where $\psi(n) = o(n)$ and $\lim_{n \rightarrow \infty} \psi = \infty$, has running time $O^*((1 + 1/\psi(n))^n) = O^*(2^{n/\psi(n)})$.

Proof. It is well-known that the complexity of a branching algorithm with branching vector $(1, \psi(n))$ is $O^*(\lambda^n)$ where λ is the positive solution of the equation:

$$X^{\psi(n)} - X^{\psi(n)-1} - 1 = 0$$

It holds that: $X^{\psi(n)} = 1/(1-1/X) \iff \psi(n) = -\log(1-1/X)/\log X$. Set $X = 1 + \varepsilon$. Then $\psi(n)$ becomes:

$$\psi(n) = \frac{-\log\left(\frac{\varepsilon}{1+\varepsilon}\right)}{\log(1+\varepsilon)} = 1 - \frac{\log(\varepsilon)}{\log(1+\varepsilon)}$$

Since $\lim_{n \rightarrow \infty} X = 1$, it holds that $\lim_{n \rightarrow \infty} \varepsilon = 0$. So, $\log(1 + \varepsilon) \sim \varepsilon$ and thus $\psi(n) \sim -\log(\varepsilon)/\varepsilon$ and:

$$\log \psi(n) \sim -\log \log(\varepsilon) - \log(\varepsilon) \sim -\log(\varepsilon)$$

Therefore, $\psi(n) \sim 1/\varepsilon$, and: $\lambda \sim 1 + 1/\psi(n)$.

Note that, since $\log(\lambda^n) = n \log(1 + 1/\psi(n)) \sim n/\psi(n) = \log(2^{n/\psi(n)})$, $\lambda^n \sim 2^{n/\psi(n)}$. \square

For simplicity, we have chosen in Lemma 1 a very simple branching vector that fits very well many optimization problems and in particular, as Lemma 2 shows, MAX INDEPENDENT SET and MIN VERTEX COVER. But the lemma works also for more general branching vectors, for instance of the form $(\psi_1(n), \psi_2(n))$.

Lemma 2. *For any $\eta > 0$, there exists an approximation preserving n^η -sparsification for MAX INDEPENDENT SET and MIN VERTEX COVER working in time $O^*(n^{1-\eta})$.*

Proof. While the maximum degree Δ of the surviving graph exceeds n^η , the standard branching has vector better than $(1, n^\eta)$ and is approximation preserving.

For MAX INDEPENDENT SET, this branching consists in either including a vertex v of maximum degree to the solution and removing $\Gamma[v]$ ($\Delta + 1$ vertices are so removed), or not including v in the solution and removing it from the graph (1 vertex removed).

For MIN VERTEX COVER, either include a vertex v of maximum degree in the solution and remove it from the graph (1 vertex removed), or discard v and mandatorily include $\Gamma(v)$ to the solution and remove $\Gamma[v]$ ($\Delta + 1$ vertices fixed).

By Lemma 1, this branching takes time $O^*(2^{n^{1-\eta}})$. \square

One of the main characteristics of the classical notions of reducibility used for proving NP-completeness (i.e., Karp- or Turing-reducibility) is the superlinear amplification of the instance sizes. This fact constitutes a major drawback for using these reductions in order to transfer (in)approximability results between problems. Most of the approximation preserving reductions (see [15] for an extensive presentation and discussion of such reductions) manage to limit this amplification in such a way that, in most cases, it remains (almost) linear. In this sense, a reduction which transforms a graph G of order n into an instance of size $O(m)$, has very few chances to be approximation preserving (the bounded-degree requirement of the L-reductions in [16] basically guarantees that m remains linear in n).

As we show in the following Theorem 3, allowing the approximation preserving sparsifier to stop before the degree becomes a constant, enables us to exploit approximation preserving reductions amplifying the instance “more than linearly”, and more precisely in $O(n+m)$. Note that, for short, the theorem handles approximability preserving reductions from Π to Π' that transform some ratio r' for Π' into ratio $r = c(r') = r'$ for Π , i.e., c is the identity function.

Theorem 3. *Under ETH:*

1. *if there exists an approximation preserving reduction from MAX INDEPENDENT SET to a problem Π building instances of size $O(n+m)$, then, for any $\varepsilon > 0$, and any r larger than some constant satisfying $r \leq n^{1/2-\varepsilon}$, Π cannot be $c(r)$ -approximable in time $O^*(2^{n^{1-2\varepsilon}/r^{1+\varepsilon}})$;*
2. *if there exists an approximation preserving reduction from MIN VERTEX COVER to a problem Π building instances of size $O(n+m)$, then, for any $\varepsilon > 0$, Π is not $c(7/6 - \varepsilon)$ -approximable in time $O^*(2^{n^{1-\varepsilon}})$.*

Proof. We first handle the case of reductions from MAX INDEPENDENT SET. For any ε , take $\eta = \varepsilon$ and apply Lemma 2 to obtain n^ε -sparse instances in time $O^*(2^{n^{1-\varepsilon}})$. Reduce all those instances to Π ; instances of size $O(n + nn^\varepsilon) = O(n^{1+\varepsilon})$ are so built. By [13], MAX INDEPENDENT SET is not r -approximable in $O^*(2^{n^{(1-\varepsilon)/(1+\varepsilon)}/r^{1+\varepsilon}})$. Thus, Π is not r -approximable in $O^*(2^{n^{1-2\varepsilon}/r^{1+\varepsilon}})$, since $(1 - 2\varepsilon)(1 + \varepsilon) = 1 - \varepsilon - 2\varepsilon^2 = 1 - \varepsilon - o(\varepsilon)$.

We now handle reductions from MIN VERTEX COVER. Beforehand let us do the following important remark. The instance of MAX INDEPENDENT SET built in [13] to ensure the inapproximability gap for MAX INDEPENDENT SET, cannot be used to produce some gap for MIN VERTEX COVER that is greater than $7/6$, the gap of Item 2 of Theorem 1 [11]. Indeed, using this instance, the negative result that can be derived for MIN VERTEX COVER

is just the impossibility of a subexponential time approximation schema. So, in what follows the gap-preserving reductions from MIN VERTEX COVER we will use the gap $7/6$ of Theorem 1.

Suppose that Π is $(7/6 - \varepsilon)$ -approximable in time $O^*(2^{n^{1-\varepsilon}})$ for some $\varepsilon > 0$. Apply Lemma 2 with $\eta = \varepsilon$ to obtain n^ε -sparse instances in time $O^*(2^{n^{1-\varepsilon}})$. Reduce all those instances to Π ; $2^{n^{1-\varepsilon}}$ instances of size $O(n + nn^\varepsilon) = O(n^{1+\varepsilon})$ are so built. By assumption, in time $2^{n^{1-\varepsilon}} 2^{(n^{1+\varepsilon})^{1-\varepsilon}} = 2^{n^{1-\varepsilon} + n^{1-\varepsilon^2}} = O(2^{n^{1-\varepsilon'}})$ (by setting, say, $\varepsilon' = 2\varepsilon^2$), one can $(7/6 - \varepsilon)$ -approximate all those subinstances and therefore one can $(7/6 - \varepsilon)$ -approximate MIN VERTEX COVER, a contradiction with Item 2 of Theorem 1. \square

3.2 A k -step sparsifier for maximization subset graph-problems

The superlinear sparsifier developped in Section 3.1 obviously works in superpolynomial time. In what follows, we develop, simple approximability preserving sparsifier, working in polynomial time. Here also, sparsification is done with respect to the maximum degree Δ of the input graph G .

We deal with maximization graph problems where feasible solutions are subsets of the vertex-set verifying some specific property (in this paper we consider hereditary property); we call informally these problems “subset problems”. Furthermore, we suppose that non-trivial feasible solutions dominate the rest of vertices of the graph. The degree decreasing (sparsification) is done thanks to this domination characteristic of the solution. For reasons of simplicity, we describe the sparsifier for the case of MAX INDEPENDENT SET, but it can be identically applied for any subset problem whose non-trivial solutions dominate the rest of the vertices of the input graph.

Consider a graph G with degree Δ and a constant $k < \Delta$. Then the sparsifier, builds an instance of MAX INDEPENDENT SET- $\Delta - k$ running the following procedure:

*for $1 \leq i \leq k$, repeatedly excavate maximal (for inclusion) independent sets X_i ,
until the degree of the surviving graph becomes equal to $\Delta - k$.*

Denote by $G'(V', E')$ the instance of MAX INDEPENDENT SET- $\Delta - k$, so-built. Note that, since maximal independent sets dominate the vertices of the graph where they are excavated, their removal reduces the maximum degree. Hence, at the end of the sparsification, G' has degree $\Delta - k$. Furthermore, the sparsifier iterates k times, that is polynomial in n .

Remark that non-trivial solutions of several maximization subset graph-problems verify vertex-domination property. This is the case, for instance of MAX ℓ -COLORABLE INDUCED SUBGRAPH, or of MAX INDUCED PLANAR SUBGRAPH. Indeed if there exists a vertex non dominated by a vertex-set V' inducing an ℓ -colorable subgraph, it suffices to add it in one of the color-classes. The graph $G[V' \cup \{x\}]$ always remains ℓ -colorable. The same holds for MAX INDUCED PLANAR SUBGRAPH.

Theorem 4. *Let $\mathcal{P}(\Pi, r', \Delta - k)$ be the following property: “if problem Π is approximable within ratio r' in time $f(n)$ on $(\Delta - k)$ -sparse graphs then, on Δ -sparse graphs, it is $(r' + 1)$ -approximable in time $O(f(n) + n^2)$ ”. Then:*

1. $\mathcal{P}(\text{MAX INDEPENDENT SET}, r', \Delta - 2);$
2. $\mathcal{P}(\text{MAX } \ell\text{-COLORABLE INDUCED SUBGRAPH}, r', \Delta - \ell);$
3. $\mathcal{P}(\text{MAX INDUCED PLANAR SUBGRAPH}, r', \Delta - 1).$

Proof. Let $G(V, E)$ be a graph on n vertices with maximum degree Δ . Let S^* be a maximum independent set of G . Run the k -step sparsifier for two steps and stop it (this obviously takes polynomial time). It computes two maximal independent sets S_1 in G , and S_2 in $G[V \setminus S_1]$; $G' = G[V \setminus (S_1 \cup S_2)]$ has degree at most $\Delta - 2$. Set $B = G[S_1 \cup S_2]$, the bipartite subgraph of G induced by the union of S_1 and S_2 .

Since B is bipartite, a maximum independent set S_B^* in B can be computed in polynomial time. If $|S_B^*| \geq \alpha(G)/r$, then S_B^* is an r -approximation MAX INDEPENDENT SET in G .

Assume now that $|S_B^*| < \alpha(G)/r$ and consider the graph $G' = G[V \setminus (S_1 \cup S_2)]$. Let $S^{*'} be the part of S^* contained in G' . Since $|S_B^*| < \alpha(G)/r$, and since S_B^* has size at least equal to the size of the part of S^* that belongs to B , $|S^{*'}| > (1 - 1/r)\alpha(G)$.$

The graph G' has degree at most $\Delta - 2$, since if a vertex v has degree Δ , or $\Delta - 1$ in $G[V \setminus (S_1 \cup S_2)]$, then it has no neighbors in either S_1 , or S_2 and this contradicts the maximality of at least one of them.

Run in G' the r' -approximation algorithm (with complexity $f(n)$) assumed for $(\Delta - 2)$ -sparse graphs and denote by S' the solution returned. Since S' is an r' -approximation, $|S'| \geq |S^{*'}|/r'$, so, $|S'| > ((1 - 1/r)1/r')\alpha(G)$. The independent set S' is obviously a solution also for G and guarantees ratio $r \cdot r' / r - 1$.

Finally, take the best among independent sets S_B^* and S' as solution for G .

Equality of ratios r and $r \cdot r' / r - 1$ derives $r = r' + 1$. Since ratio r' is achieved in time $f(n)$ and the application of the sparsification step takes time $O(n^2)$, ratio r is achieved for MAX INDEPENDENT SET in G in time $O(f(n) + n^2)$ as claimed.

For MAX ℓ -COLORABLE INDUCED SUBGRAPH, let $G(V, E)$ be a graph on n vertices with maximum degree Δ . Let L^* be an optimal solution for MAX ℓ -COLORABLE INDUCED SUBGRAPH on G . Run the k -step sparsifier for MAX INDEPENDENT SET for ℓ steps. It iteratively excavates ℓ maximal independent sets S_1, S_2, \dots, S_ℓ . Set $V' = S_1 \cup S_2 \cup \dots \cup S_\ell$, and $G' = G[V']$, the ℓ -colorable subgraph of G induced by V' . Denote by $L^{*'} the part of L^* belonging to L^* .$

If $|L^{*'}| \geq L^*/r$ then, since $|V'| \geq |L^{*'}| \geq L^*/r$, V' is an r -approximation MAX INDEPENDENT SET in G .

Assume now $|L^{*'}| < L^*/r$ and consider the graph $G'' = G[V \setminus V']$. Let $L^{*''}$ be the part of L^* contained in G'' . Since $|L^{*'}| < L^*/r$, $|L^{*''}| > (1 - 1/r)|L^*|$.

The graph G'' has degree at most $\Delta - \ell$ and the rest of the proof remains similar to the corresponding part of that of the first item.

For MAX INDUCED PLANAR SUBGRAPH, one just excavates only one independent set. An independent set is a planar graph. The rest of the proof of the third item is the same as above. \square

4 Subexponential inapproximability

4.1 Via superlinear sparsification

Combining the superlinear sparsifier of Definition 1 in Section 3.1 together with approximation preserving reductions from MIN VERTEX COVER to several problems, the following theorem can be proved.

Theorem 5. *Under ETH, and for any $\varepsilon > 0$, none of MIN DOMINATING SET, MIN SET COVER and MIN HITTING SET, MIN FEEDBACK VERTEX SET, MIN INDEPENDENT DOMINATING SET, and MIN FEEDBACK ARC SET is $(7/6 - \varepsilon)$ -approximable in time $O^*(2^{n^{1-\varepsilon}})$.*

Proof. For MIN DOMINATING SET, let $G(V, E)$ be an instance of MIN VERTEX COVER and assume G is connected. Build a graph $G'(V', E')$ as follows. Start from a copy of G and for each edge $e = (u, v) \in E$, add two dummy vertices y_e and z_e in V' and link those vertices to u and v . The graph G' so built has order $n + 2m$.

A minimum dominating set in G' does not contain any dummy vertex. Indeed, if a solution S contains $y_{(u,v)}$ or $z_{(u,v)}$, then $S \setminus \{y_{(u,v)}, z_{(u,v)}\} \cup \{u\}$ is still a dominating set of at most equal cardinality. Thus, a minimum dominating set in G' naturally maps to a subset of V which covers all the edges, hence a vertex cover of the same size. Furthermore, given an r -approximation of MIN DOMINATING SET in G' , one can start by removing the potential dummy vertices as explained above, and then obtain an r -approximation for MIN VERTEX COVER. Item 2 of Theorem 3 suffices for completing the proof.

The result for MIN SET COVER immediately follows from a well-known approximation preserving reduction from MIN DOMINATING SET (function c being the identity function). Given an instance $G(V, E)$ of MIN DOMINATING SET, one can construct an instance (\mathcal{S}, C) of MIN SET COVER, where \mathcal{S} is a set-system over the ground set C , by taking $\mathcal{S} = V$, $C = V$ and, for each vertex $v_i \in V$, the corresponding set $S_i \in \mathcal{S}$ contains as elements $c_j \in C$ such that vertex v_j is either v_i or $v_j \in \Gamma(v_i)$.

For MIN HITTING SET, just observe is the problem is similar to MIN SET COVER where roles of \mathcal{S} and C are interchanged.

Notice that the previous reduction still works for MIN FEEDBACK VERTEX SET. In G' , every subset of vertices containing non-dummy vertex is a dominating set, iff it is a feedback vertex set³.

For MIN INDEPENDENT DOMINATING SET, tune the previous reduction by deleting all the edges in the copy of the graph G . In other words, build G' from an independent set V of size $n = |V|$ where each vertex corresponds to a vertex in V , and link all the vertices $u \in V$ to an independent set I_e with 2 dummy vertices for each edge $e = (u, v)$. Again, an optimal solution contains only copy vertices (no dummy vertices). Furthermore, in G' , every subset containing non-dummy vertex is an independent dominating set iff it is a vertex cover in G .

For MIN FEEDBACK ARC SET, the reduction in [17] is approximation preserving with c the identity function. The graph $G'(V', E')$ for MIN FEEDBACK ARC SET is built with:

$$\begin{aligned} V' &= V \times \{0, 1\} \\ E' &= \{((u, 0), (u, 1)) : u \in V\} \cup \{((u, 1), (v, 0)) : (u, v) \in E\} \end{aligned}$$

In any solution, an arc $((u, 1), (v, 0))$ can be advantageously replaced by arc $((v, 0), (v, 1))$. Indeed, a cycle containing edge $((u, 1), (v, 0))$, necessarily contains also edge $((v, 0), (v, 1))$ since the vertex $(v, 0)$ has out-degree 1. Thus, removing $((v, 0), (v, 1))$ destroys the same cycles (plus potentially others). We can therefore assume that a solution is $\{((v, 0), (v, 1)) : v \in S\}$, for some $S \subseteq V$. Now, S is a vertex cover, and an r -approximation for MIN FEEDBACK ARC SET transforms into an r -approximation for MIN VERTEX COVER. \square

Let us note that using the classical reduction from MIN VERTEX COVER to MIN SAT [18] a similar result can be derived for MIN SAT.

4.2 Via k -step sparsification

Revisit Item 2 of Theorem 2. There, Δ_ε is related to ε in the following way: there exists a universal constant C such that $\Delta_\varepsilon = 2^{C/\varepsilon}$. Our purpose in this section is to

³These reductions rely on the fact that, in graphs without isolated vertices, a vertex cover is both a dominating set and a feedback vertex set.

strengthen this item deriving inapproximability for MAX INDEPENDENT SET, MAX ℓ -COLORABLE INDUCED SUBGRAPH and MAX INDUCED PLANAR SUBGRAPH, in subexponential time $O^*(2^{O(n^{1-\varepsilon}/\Delta_\varepsilon^{1+\varepsilon})})$ with a *smaller* bounded degree.

Theorem 6. *Under ETH, for any $\varepsilon > 0$ and any $\Delta < \Delta_\varepsilon$, in Δ -sparse graphs, MAX INDEPENDENT SET, MAX ℓ -COLORABLE INDUCED SUBGRAPH and MAX INDUCED PLANAR SUBGRAPH are inapproximable within ratios $\Delta/2 - (\Delta_\varepsilon/2 - \Delta_\varepsilon^{1-\varepsilon})$, $\Delta/2 - (\Delta_\varepsilon/\ell - \Delta_\varepsilon^{1-\varepsilon})$ and $\Delta - (\Delta_\varepsilon - \Delta_\varepsilon^{1-\varepsilon})$, respectively, in time $O^*(2^{O(n^{1-\varepsilon}/\Delta_\varepsilon^{1+\varepsilon})})$.*

Proof. By Item 2 of Theorem 2, for any $\varepsilon > 0$, MAX INDEPENDENT SET on Δ_ε -sparse graphs, is inapproximable within ratio $\Delta_\varepsilon^{1-\varepsilon}$ in time $O^*(2^{n^{1-\varepsilon}/\Delta_\varepsilon^{1+\varepsilon}})$, with $\Delta_\varepsilon = 2^{C/\varepsilon}$ for some constant C .

For any Δ , run the k -step sparsifier on a Δ_ε -sparse graph G for $(\Delta_\varepsilon - \Delta)/2$ steps, from Δ_ε down to Δ , in order to get a Δ -sparse instance G' of MAX INDEPENDENT SET. Combination of the Item refmain1 of Theorem 4 and of Item 2 of Theorem 2 directly derives inapproximability of MAX INDEPENDENT SET in G within ratio $\Delta_\varepsilon^{1-\varepsilon} - (\Delta_\varepsilon - \Delta)/2 = \Delta/2 - (\Delta_\varepsilon/2 - \Delta_\varepsilon^{1-\varepsilon})$ in time $O^*(2^{O(n^{1-\varepsilon}/\Delta_\varepsilon^{1+\varepsilon})})$.

Consider now the following simple reduction from MAX INDEPENDENT SET to MAX ℓ -COLORABLE INDUCED SUBGRAPH. Let $G(V, E)$ be an instance of MAX INDEPENDENT SET of order n . We keep G as the instance of MAX ℓ -COLORABLE INDUCED SUBGRAPH. Any independent set S of G can be considered as an ℓ -colorable graph with empty the $\ell - 1$ of its color classes. Conversely, given an ℓ -colorable graph on sets S_1, S_2, \dots, S_ℓ , all them are independent sets and the largest among them has size more than $1/\ell$ times the size of the ℓ -colorable graph. So, any ratio r for MAX ℓ -COLORABLE INDUCED SUBGRAPH becomes ratio ℓr for MAX INDEPENDENT SET.

In the same spirit, one can devise a reduction from MAX INDEPENDENT SET to MAX INDUCED PLANAR SUBGRAPH. An independent set is a planar graph per se. On the other hand since any planar graph is 4-colorable, a solution $G' = G[S]$ of MAX INDUCED PLANAR SUBGRAPH can be transformed into an independent set by coloring the vertices of S with four colors and taking the largest of them. So an approximation ratio r for MAX INDUCED PLANAR SUBGRAPH is transformed into ratio $4r$ for MAX INDEPENDENT SET.

The proofs for MAX ℓ -COLORABLE INDUCED SUBGRAPH and MAX INDUCED PLANAR SUBGRAPH above of the theorem immediately derive from the remarks above. \square

Note that the inapproximability bound for MAX INDEPENDENT SET of Theorem 6 (Item 1) cannot be derived by Theorem 2 for $\Delta \geq 2^{C/\varepsilon}(1/2 - 2^{-C})$. So, Theorem 6 extends the result of [13] to degree $\Delta_\varepsilon/2$.

Also, from the discussion of for MAX ℓ -COLORABLE INDUCED SUBGRAPH and MAX INDUCED PLANAR SUBGRAPH in the proof of Theorem 6, the following corollary holds.

Corollary 1. *Under ETH, and for any $\varepsilon > 0$, neither MAX ℓ -COLORABLE INDUCED SUBGRAPH nor MAX INDUCED PLANAR SUBGRAPH is r -approximable in time $O^*(2^{n^{1-\delta/r^{1+\delta}}})$, where r is the approximability-gap of MAX INDEPENDENT SET.*

4.3 Via Theorem 2

Similar results as those of Corollary 1 can be obtained for several other problems linked to MAX INDEPENDENT SET by approximability-preserving reductions.

For instance, for MAX SET PACKING, take $\mathcal{S} = V$, $C = E$ and, for any set $S_i \in \mathcal{S}$, $S_i = \{e_j : e_j \text{ incident to } v_i\}$. This very classical reduction transforms any independent set of G to an equal-cardinality set-packing of (\mathcal{S}, C) , and vice-versa.

For MAX UNUSED SETS, observe that its optimal value is an affine transformation of the optimum for MIN SET COVER. Since this latter problem is a generalization of MIN VERTEX COVER (indeed MIN VERTEX COVER can be seen as a MIN SET COVER problem where all ground elements have frequency 2), MAX UNUSED SETS is a generalization of MAX INDEPENDENT SET.

In what follows in this section, we handle inapproximability bounds for two problems that are closely linked between them, MIN INDEPENDENT DOMINATING SET and MAX MINIMAL VERTEX COVER. In fact, they are related in the same way as MAX INDEPENDENT SET and MIN VERTEX COVER.

Let us first consider MAX MINIMAL VERTEX COVER and revisit the following reduction from MAX INDEPENDENT SET given in [19]. Given an instance $G(V, E)$ of MAX INDEPENDENT SET, link any $v_i \in V$ to $n + 1$ new vertices. The so-built graph H for MAX MINIMAL VERTEX COVER has size $n^2 + 2n$. Then, by considering a MAX MINIMAL VERTEX COVER-solution for H consisting of taking the out-of- G neighbors of some independent set S of G together with $V \setminus S$ as solution for MAX MINIMAL VERTEX COVER, one can guarantee the following:

$$\begin{aligned} \text{sol}(H) &\leq n \cdot |S| + n \\ \text{opt}(H) &\geq n \cdot \alpha(G) + n \end{aligned} \tag{1}$$

where $\text{sol}(H)$ and $\text{opt}(H)$ denote the sizes of an approximate and of an optimal solutions for MAX MINIMAL VERTEX COVER, respectively. Then, using expressions in (1) and considering G the MAX INDEPENDENT SET-instance of [13], one easily derives the following.

Proposition 1. *Under ETH, for any $\delta > 0$ and any $r \geq n^{1/4-\delta}$, MAX MINIMAL VERTEX COVER is inapproximable within ratio r in less than $O^*(2^{n^{1/2-\delta/r^{1+\delta}}})$ time.*

Observe that, in the reduction above, $\Delta(H) \geq n \approx \sqrt{n(H)}$. So, the following corollary derives from Proposition 1.

Corollary 2. *Furthermore, under ETH, for any $\delta > 0$ and any $r \geq \Delta^{1/2-\delta}$, MAX MINIMAL VERTEX COVER is inapproximable within ratio r in less than $O^*(2^{\Delta^{1-\delta/r^{1+\delta}}})$ time.*

The result of Proposition 1 can be further strengthened by slightly changing the reduction of [19]. Denote by c the stability ratio $\alpha(G)/n$ of G . Then the following holds.

Proposition 2. *Under ETH, for any $\delta > 0$ and any $r \geq n^{1/2-\delta}$, in any graph of order nr , MAX MINIMAL VERTEX COVER is inapproximable within ratio $(c+r)/(1+c)$ in time less than $O^*(2^{n^{1-\delta/r^{1+\delta}}})$, where c the stability ratio of the MAX INDEPENDENT SET-instance of [13].*

Proof. Consider the MAX INDEPENDENT SET-instance of Theorem 2 and link any of its vertices to $r + 1$ new vertices where r is as in Item 1 of Theorem 2. The MAX MINIMAL VERTEX COVER-instance H has now $n(r + 1)$ vertices. Set $\rho'(H) = \text{sol}(H)/\text{opt}(H)$, the inverse of the approximation ratio for MAX MINIMAL VERTEX COVER in H . Then, using (1), it holds that:

$$\frac{|S|}{\alpha(G)} \geq \rho'(H) - \frac{(1 - \rho'(H))n}{r\alpha(G)} \tag{2}$$

As one can see in the proof of Item 1 of Theorem 2, $\alpha(G)$ is linear in n , i.e., $\alpha(G) \geq cn$ for some fixed (independent on n) $c < 1$. So, (2) becomes:

$$\frac{1}{r} \geq \frac{|S|}{\alpha(G)} \geq \rho'(H) - \frac{(1 - \rho'(H))c}{r} \geq \rho'(H) - \frac{c}{r} \tag{3}$$

where the first inequality above is due to the inapproximability bound $1/r$ for MAX INDEPENDENT SET in the graph of Item 1 of Theorem 2. Then some simple algebra derives $\rho(H) = 1/\rho'(H) \geq c+r/1+c$, as claimed. \square

Interestingly enough, although MIN INDEPENDENT DOMINATING SET is one of the hardest problems for polynomial approximation, only subexponential inapproximability within ratio $7/6 - \varepsilon$ can be proved for it, using sparsification. The following proposition gives a stronger subexponential inapproximability bound for MIN INDEPENDENT DOMINATING SET using the fact that an independent dominating set in some graph G is the complement of a minimal vertex cover of G .

Proposition 3. *Under ETH, for any $\delta > 0$ and any $r \geq n^{1/2-\delta}$, in any graph of order nr , MIN INDEPENDENT DOMINATING SET is inapproximable within ratio $1/(1-c)$ in time less than $O^*(2^{n^{1-\delta/r^{1+\delta}}})$, where c is the stability ratio $\alpha(G)/n$ of the MAX INDEPENDENT SET-instance of [13].*

Proof. Consider again the graph G built in Item 1 of Theorem 2 and the reduction of Proposition 2 to MAX MINIMAL VERTEX COVER. Denote by c the stability ratio of G , i.e., $c = \alpha(G)/n$, and recall that c is a fixed constant [13]. Then:

$$\iota(H) = \alpha(G) + (n - \alpha(G))(r + 1) = (1 - c)n(r + 1) \quad (4)$$

Denote by $\iota'(H)$, the independent dominating set associated with the approximate minimal vertex cover of H , i.e., $\iota'(H) = n(r + 1) - \text{sol}(H)$ and by b the inverse of the inapproximability bound for MAX MINIMAL VERTEX COVER ($b < 1$). Then, using (4), we get:

$$\begin{aligned} b \geq \frac{\text{sol}(H)}{\text{opt}(H)} &\geq \frac{n(r + 1) - \iota'(H)}{n(r + 1) - \iota(H)} \implies \frac{\iota'(H)}{\iota(H)} \geq \frac{n(r + 1)(1 - b)}{\iota(H)} + b \\ &\geq \frac{1 - b}{1 - c} + b \sim \frac{1}{1 - c} \end{aligned}$$

where the last approximation for b is due to the fact that $b = o(1)$. \square

5 More about sparsifiers

Revisit the informal description of sparsification in Section 3.1. The sparsifier designed in [12] may yield very weak lower bounds, in the sense that $f(\lambda)$ may be very close to 1. Suppose that there exists a polynomial time reduction R from k -SAT to a problem Π , and two integers α and β such that, for an instance ϕ of k -SAT with n variables and m clauses, $R(\phi)$ is of size $\alpha n + \beta m$. To solve an instance of k -SAT on ϕ , one can sparsify it, reduce all the $2^{\varepsilon n}$ sparsified formulæ, and solve each instance of Π built by application of R to any sparse instance produced from ϕ . This takes time $O^*((2^{\varepsilon} \lambda^{\alpha + \beta C_k^{\varepsilon}})^n)$. Assuming ETH, let λ_k be the smallest real number such that k -SAT is solvable in $O^*(\lambda_k^n)$. Then, $2^{\varepsilon} \lambda^{\alpha + \beta C_k^{\varepsilon}} \geq \lambda_k$. Adjusting ε to get the best possible lower bound for λ , one gets $\lambda - 1 < 10^{-10}$, for plausible values of α and β . So, one only shows that Π is not solvable in, say, $O^*((1 + 10^{-10})^n)$.

We show that the superlinear sparsifier of Section 3.1 may be used to produce stronger lower bounds than those get by the sparsifier of [12]. In order to do that, we will use the central problem of the paper, the MAX INDEPENDENT SET problem. Assume $\mathcal{H}_{\text{IS}}(\lambda)$ is the hypothesis that MAX INDEPENDENT SET is not solvable in time $O^*(\lambda^n)$, and $g : (1, 2) \rightarrow \mathbb{N}$ maps any real value x in $(1, 2)$ to the smallest integer p such that the positive root $X^{p+1} - X^p - 1 = 0$ is smaller than x . The superlinear sparsifier can be used to show the following.

Proposition 4. *Let Π be problem such that there exists a polynomial time reduction R from MAX INDEPENDENT SET to Π and two positive numbers α and β satisfying, for all instances G of MAX INDEPENDENT SET, $|R(G(V, E))| \leq \alpha|V| + \beta|E| = \alpha n + \beta m$. Under $\mathcal{H}_{IS}(\lambda)$, if Π is solvable in $O^*(\mu^n)$, then $\mu > \lambda^{1/\alpha + \lfloor g(\lambda)/2 \rfloor \beta}$*

Proof. Use the superlinear sparsifier with the threshold $\Delta = g(\lambda)$, that is, stop the branching when the degree of the graph becomes strictly less than $g(\lambda)$. The branching factor is the positive root of $X^{g(\lambda)+1} - X^{g(\lambda)} - 1 = 0$ which, by construction, is smaller than λ . At a leaf of the branching tree, if the number of vertices is $n - k$, then the number of edges in the remaining graph is at most $\lfloor g(\lambda)/2 \rfloor (n - k)$.

Thus, by performing the reduction R on the instances at each leaf of the branching tree, and then solving the obtained instances of Π , one gets an algorithm solving MAX INDEPENDENT SET in time $O^*(\lambda^k \mu^{(\alpha + \lfloor g(\lambda)/2 \rfloor \beta)(n-k)})$. So, $\mu > \lambda^{1/\alpha + \lfloor g(\lambda)/2 \rfloor \beta}$, otherwise $\lambda^k \mu^{(\alpha + \lfloor g(\lambda)/2 \rfloor \beta)(n-k)} \leq \lambda^n$. \square

Since the superlinear sparsifier is approximation preserving, if reduction R from MAX INDEPENDENT SET to Π preserves approximation, one can obtain relative exponential time lower bounds even for approximation issues. The following proposition provides a lower bound to the best currently known complexity (function of the number of clauses) of MAX 3-SAT, under \mathcal{H}_{IS} . Note that the best known running time for MAX 3-SAT is $O^*(1.324^m)$.

Proposition 5. *Under $\mathcal{H}_{IS}(\lambda)$, MAX 3-SAT is not solvable in $O^*(\lambda^{(1/1 + \lfloor g(\lambda)/2 \rfloor)n})$.*

Proof. We recall the reduction in [16]. An instance $(G(V, E), k)$ of the decision version of MAX INDEPENDENT SET is transformed into an instance of the decision version of MAX 3-SAT in the following way: each vertex $v_i \in V$ encodes a variable X_i and for each edge $(v_i, v_j) \in E$ we add a clause $\neg X_i \vee \neg X_j$. Finally, we add the 1-clause X_i for all $v_i \in V$. In the so built instance of MAX 3-SAT we wish to satisfy at least $k + m$ clauses. This reduction builds $n + m$ clauses, so $\alpha = \beta = 1$. Hence, under \mathcal{H}_{IS} , and according to Proposition 4, one cannot solve MAX 3-SAT in time $O^*(\mu^n)$ when $\mu = \lambda^{1/1 + \lfloor g(\lambda)/2 \rfloor}$. \square

Suppose that Π is a problem (like MAX 3-SAT when considering its complexity in terms of m) with a reduction from MAX INDEPENDENT SET in $n + m$ ($\alpha = \beta = 1$), and Π is solvable in $O^*(\mu^n)$. Then, the following table gives some values of μ as function of λ .

λ	Infeasible value for μ
1.1	1.0073
1.18	1.027
1.21	1.038

We conclude the paper by pointing out that the k -step sparsifier of Section 3.2 has also some interesting consequences when handling parameterized issues. MAX INDEPENDENT SET can be solved in time $O^*((\Delta + 1)^\alpha)$ with a standard branching algorithm [20] (here $\alpha = \alpha(G)$ is the size of a maximum independent set, or equivalently the natural parameter for MAX INDEPENDENT SET). The excavation performed by the k -step sparsifier can be used to obtain an algorithm running in time $O^*(2^{(\Delta-2)\alpha})$. Indeed, one can excavate consecutively $\Delta - 2$ maximal independent sets S_1 to $S_{\Delta-2}$, where each S_i is a maximal independent set in $G[V \setminus \bigcup_{k=1 \dots i-1} S_k]$. By hypothesis, for all i , $|S_i| \leq \alpha$, so an exhaustive search on $\bigcup_{k=1 \dots \Delta-2} S_i$ takes time $O^*(2^{(\Delta-2)\alpha})$. Graph $G[V \setminus \bigcup_{k=1 \dots \Delta-2} S_k]$ is a graph with degree 2, hence it takes polynomial time to complete a solution by finding a maximum independent set on this part of the graph. This algorithm improves the branching algorithm for $\Delta \leq 4$, as the following table shows.

Δ	Exhaustive branching	Sparsification
3	4^α	2^α
4	5^α	4^α

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A Definition of the problems handled in the paper

- **MAX INDEPENDENT SET.** Given a graph $G(V, E)$, determine a maximum cardinality set $V' \subseteq V$, such that any two vertices of V' are not adjacent in G .
- **MIN VERTEX COVER.** Given a graph $G(V, E)$, determine a minimum cardinality set $V' \subseteq V$, such that any edge in E has at least one of its endpoints in V' .
- **MIN DOMINATING SET.** Given a graph $G(V, E)$, determine a minimum cardinality set $V' \subseteq V$ such that every vertex $v \in V \setminus V'$ is neighbor of some vertex in V' .
- **MIN INDEPENDENT DOMINATING SET.** Given a graph $G(V, E)$, determine a minimum cardinality set $V' \subseteq V$ that is simultaneously an independent and a dominating set.
- **MIN FEEDBACK VERTEX SET.** Given a graph $G(V, E)$, determine a minimum cardinality set $V' \subseteq V$, such that any cycle of G has at least one vertex in V' .
- **MAX COMPLETE BIPARTITE SUBGRAPH.** Given a graph $G(V, E)$, determine a maximum cardinality set $V' \subseteq V$ that induces a complete bipartite graph.
- **MAX ℓ -COLORABLE INDUCED SUBGRAPH.** Given a graph $G(V, E)$ and some fixed constant ℓ , determine a maximum cardinality set $V' \subseteq V$ that induces an ℓ -colorable graph.
- **MAX PLANAR INDUCED SUBGRAPH.** Given a graph $G(V, E)$, determine a maximum cardinality set $V' \subseteq V$ that induces a planar graph.
- **MIN SET COVER.** Given a system \mathcal{S} of subsets of a ground set C , determine a minimum cardinality subsystem \mathcal{S}' that covers C .
- **MIN HITTING SET.** Given a system \mathcal{S} of subsets of a ground set C , determine a minimum cardinality subset $C' \subseteq C$ that hits all the sets of \mathcal{S}' .
- **MAX SET PACKING.** Given a system \mathcal{S} of subsets of a ground set C , determine a maximum cardinality subsystem \mathcal{S}' of pairwise disjoint sets.
- **MAX MINIMAL VERTEX COVER.** Given a graph $G(V, E)$, determine a maximum cardinality set $V' \subseteq V$, that is a minimal (for exclusion) vertex cover of G .
- **MAX UNUSED SETS.** Given a system \mathcal{S} of subsets of a ground set C , determine a maximum cardinality subsystem \mathcal{S}' such that $\mathcal{S} \setminus \mathcal{S}'$ covers C .
- **MIN FEEDBACK ARC SET.** Given a directed graph $G(V, E)$, determine a minimum cardinality set $E' \subseteq E$, such that any cycle of G has at least one edge in E' .